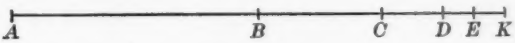
the latter the greater; and, finally, the attributes equal, greater, and less are not applicable to infinite but solely to finite quantities."

We shall see that Galileo has been curiously misinterpreted by some writers, including Cauchy, as demonstrating here that an actual infinity has no existence. That there should be as many squares as there are integers altogether was taken as absurd; hence the existence of actual infinity was considered disproved. Galileo's skill in the use of the infinite in demonstrations is shown in the following passage on falling bodies:[1]

"If the velocity were proportional to the distance through which it has fallen or is to fall, then those distances would be passed over in equal times; thus, if the velocity with which a body overcomes four yards is to be double the velocity with which the first two yards were overcome, then the times needed for these two processes would be the same; but four yards can be overcome in the same time as two yards only in the case of instantaneous motion; we see on the contrary that the body needs time to fall, and that it needs less time for a fall of two yards than of four yards; hence it is not true that the velocities increase proportionally to the distance fallen."

Gregory St. Vincent. The most important discussion of Zeno given at this time is that by Gregory St. Vincent, in his *Opus geometricum quadraturæ circuli et sectionum coni*, published at Antwerp in 1647, but written apparently twenty-five years earlier. It is a massive volume of 1400 pages. Influenced in his geometrical researches by the medieval scholastic concept of the continuum, according to which a line divided repeatedly is not reduced to indivisible elements as taught by the atomists, but admits of being subdivided *ad infinitum*, Gregory St. Vincent took a step different from that of Archimedes. While, in his proofs, Archimedes kept on dividing, only until a certain degree of smallness was reached, St. Vincent permitted the subdivisions to continue *ad infinitum*. Using unlimited section in geometry he introduced a geometric series that was truly an *infinite* series.[2]



This much had been accomplished by at least one writer before him,[3] but, so far as now known, he is the first to apply the infinite geometric progression to the study of the "Achilles." Taking a definite line segment AK he divides it at B in a given ratio, then he divides BK in the same ratio at C, and so on. The segments AB, BC, CD, . . . form an infinite geometric progression. The points C, D, E . . . lie, all of them, between A and K; they approach K as near as we please, but (in accordance with scholastic philosophy) never reach it. As Gregory conceives this matter, K is an obstacle, so to speak, against the further advance of the series of points A, B, C, . . . , similar to a rigid wall. "Terminus progressionis est seriei finis, ad quem nulla progressio pertinget, *licet* in infinitum continuetur; sed quovis interuallo dato proprius ad eum *accedere* poterit." By "series" is meant the segment AK, by "progressio," the segments AB, AC, ...

[1] *Ostwald's Klassiker*, No. 24, p. 17.

[2] Gregory St. Vincent, *Opus geometricum*, T. 1, pp. 51–56, 95–97; for our knowledge of this part of the book we are dependent entirely upon C. R. Wallner's account in the *Bibliotheca mathematica*, 3. F., Vol. IV, 1903, pp. 251–255.

[3] See H. Wieleitner in *Bibliotheca mathematica*, 3. F., Vol. 14, 1914, pp. 150–168.

Gregory states his conclusion thus: "Dico magnitudinem *AK* aequalem esse toti progressioni magnitudinum continue proportionalium, rationis *AB* ad *BC* in infinitum continuatae; siue quod idem est, rationis *AB* ad *BC* in infinitum continuatae terminum esse *K*." Considering the "Achilles" in this connection, he associates this paradox on motion for the first time definitely with the summation of an infinite series. Moreover, Gregory St. Vincent is the first writer known to us who states the exact time and place of overtaking the tortoise. So far as we are able to ascertain, Gregory was not troubled, in explaining the "Achilles," by the fact that in his theory, the variable does not *reach* its limit. Nor, apparently, did this matter trouble his readers. His mode of solving the problem appealed to many. We shall see that Leibniz makes special reference to it. Over a century after Gregory's publication, Saverien refers in his dictionary[1] to the "Achilles," "dont Gregoire de Saint Vincent a fait voir la fausseté." Formey gave Gregory St. Vincent's explanation in the article "Mouvement" in Diderot's *Encyclopédie* (1754), later reprinted in the *Encyclopédie méthodique*, and in 1800 translated at Padova into the Italian language. The definition of a limit as given in the *Encyclopédie méthodique* does not allow the variable to surpass its limit but places no obstacle in the way of its reaching its limit.

Descartes, De Morgan and Others. Descartes at one time discussed the "Achilles." His treatment is much like that of Gregory St. Vincent. It is given in a letter of July, 1646, to Clerselier.[2] He lets Achilles, or in his place a horse, be, at the start, 10 leagues behind the tortoise, but moving ten times more rapidly than the latter. The real difficulty of the paradox he does not touch, for he says:

> "L'Achille de Zenon ne sera pas difficile à soudre, si on prend garde que, si à la dixiéme partie de quelque quantité on adioute la dixiéme de cette dixiéme, qui est une centiéme, & encore la dixiéme de cette derniere, qui n'est qu'une milliesme de la premiere, & ainsi à l'infini, toutes ces dixiémes jointes ensemble, quoy qu'elles soient suposées réellement infinies, ne composent toutesfois qu' une quantité finie, sçauvoir une neusiéme de la premiere quantité . . . Et la caption est en ce qu'on imagine que cette neusiéme partie d'une lieuë est une quantité infinie, à cause qu'on la divise par son imagination en des parties infinies."

Descartes looked upon the actually infinite as mysterious, but not impossible or absurd. He seemed to accept it in the abstract, but deny it in the concrete. At this time and even earlier (see the foregoing extracts from Galileo) there was talk about the finitude of the human mind and its consequent inability to conceive the infinite. This was ridiculed by De Morgan. He claimed that if the human mind is limited, we tacitly postulate the "unknowable"; moreover, even if the human mind were finite, there is no more reason against its conceiving the infinite than there is for a mind to be blue in order to conceive of a pair of blue eyes. Or, as De Morgan puts it in another place, the argument amounts to this, "who drives fat oxen should himself be fat." From Descartes to Hamilton,

[1] Saverien, *Dictionnaire universel de mathèmatique et de physique*, Paris, 1753, Art. "Mouvement."

[2] *Oeuvres de Descartes* par Charles Adam et Paul Tannery, T. IV, pp. 445–447.

says De Morgan,[1] this doctrine is accepted by many minds. But its genesis is found, as we have stated, long before Descartes.

A wholly different, but no more satisfactory explanation of "Achilles" comes from another Frenchman of that time, Pierre Gassendi, the physicist. In his view Zeno's proofs need no refutation, if with Epicurus one assumes not points but atoms. A difficulty seems to arise from differences in velocity of motion, for in the same time that a body moves over the physically indivisible, the more rapid body must travel over several indivisibles. In his opinion this difficulty may perhaps be overcome by conceiving motion as discontinuous, and slower motion as a mixture of rest and motion. To the senses motion would still seem continuous.[2] To those who experienced difficulties in accepting the existence of indivisible atoms, the capuchin, Casimir of Toulouse, offers an easy solution by reminding that angels had extension, yet were physically indivisible.[3]

It is worthy of note that John Dee, the famous astrologer who wrote an elaborate mathematical preface to Billingsley's edition of Euclid (1570), departs from the contention that two lines containing the same number of parts must be of equal length. He says:

"Our least Magnitudes can be divided into so many partes as the greatest. As, a Line of an inch long (with vs) may be divided into as many partes, as may the diameter of the whole world, from East to West: or any way extended."

Discussion of Thomas Hobbes. The earliest British writer, after Duns Scotus, to take up explicitly Zeno's arguments is the philosopher, Thomas Hobbes (1588–1679). In 1655 he wrote:[4]

". . . the force of that famous argument of Zeno against motion, consisted in this proposition, *whatsoever may be divided into parts, infinite in number, the same is infinite;* which he without doubt, thought to be true, yet nevertheless is false. For to be divided into infinite parts, is nothing else but to be divided into as many parts as any man will. But it is not necessary that a line should have parts infinite in number, or be infinite, because I can divide and subdivide it as often as I please; for how many parts soever I make, yet their number is finite; because he that says parts, simply, without adding how many, does not limit any number, but leaves it to the determination of the hearer, therefore we say commonly, a line may be divided infinitely; which cannot be true in any other sense."

With Hobbes, *infinite* is synonymous with *indefinite*. He takes an agnostic attitude toward problems of infinity:

"But when no more is said than this, *number is infinite*, it is to be understood as if it were said, this name *number* is an *indefinite* name. . . . And, therefore, that which is commonly said, that space and time may be divided infinitely, is not to be so understood, as if there might be any infinite or eternal division; but rather to be taken in this sense, *whatever is divided is divided into such parts as may again be divided*. . . . Who can commend him that demonstrates thus? 'If the world be eternal, then an infinite number of days, or other measures of time, preceded the birth of Abraham. But the birth of Abraham preceded the birth of Isaac; and therefore one

[1] A. De Morgan, "On Infinity; and on the Sign of Equality," in Trans. of the Cambridge Philosoph. Society, Vol. XI, p. 157, Cambridge, 1871 [read May 16, 1864].

[2] Gassendi, *Opera omnia*, 1658, I, p. 300a. An. I, p. 239; Lasswitz, *op. cit.*, Vol. II, p. 150.

[3] Lasswitz, *op. cit.*, Vol. II, p. 494.

[4] *The English Works of Thomas Hobbes*, Vol. I, London, 1839, pp. 63, 64, 413. Hobbes refers to Zeno's arguments also in his Latin works. See Thomae Hobbes, *Opera philosophica*, Vol. V, Londini, 1845, pp. 207–213.

infinite is greater than another infinite, or one eternal than another eternal; which,' he says, 'is absurd.' This demonstration is like his, who from this, that the number of even numbers is infinite, would conclude that there are as many even numbers as there are numbers simply, that is to say, the even numbers are as many as all the even and odd together. They which in this manner take away eternity from the world, do they not by the same means take away eternity from the creator of the world? . . . And the men that reason thus absurdly are not idiots, but, which makes this absurdity unpardonable, geometricians, and such as take upon them to be judges, impertinent, but severe judges of other men's demonstrations."

The reference to odd and even numbers doubtless arose from his contact with Galilean thought. While sojourning on the Continent, he had gone to see Galileo, then a prisoner. Hobbes thought he had effected the duplication of the cube and the squaring of the circle. On this matter he became involved in a heated controversy with the algebraist, John Wallis. The aged Hobbes was no match against young Wallis on mathematical questions. When the mathematical works of Wallis were being brought out, Wallis refused to allow his controversial matter against Hobbes to be incorporated in them.[1] Whether the whole is greater than a part was an issue touched upon during this dispute. Hobbes said to Wallis: "All this arguing of infinities is but the ambition of school boys." It cannot be said that Hobbes made any real contribution to a deeper understanding of the "Achilles" or any of Zeno's other arguments on motion. His objection to the dictum, "whatever may be divided into parts infinite in number, the same is infinite," is no new contribution; Aristotle had advanced that far. How Achilles caught the turtle is beyond comprehension through our sensual imagination; Hobbes nowhere explains this inability. However, he does touch upon the concept of a limit in his controversy with Wallis. Hobbes charged that some of the principles of the professors are "void of sense"; one of those principles being, "that a quantity may grow less and less eternally, so as at last to be equal to another quantity; or, which is all one, that there is a last in eternity."[2]

A GENERAL FORMULA FOR THE VALUATION OF SECURITIES.[3]

By JAMES W. GLOVER, University of Michigan.

The object of this paper is to derive a formula for the valuation of a very general type of securities. The security is redeemed in r equal installments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m installments, and the security is purchased to realize the investor a nominal rate of interest j with frequency of conversion m.

[1] A full account of the controversy between Hobbes and Wallis is given in Croom Robertson's *Hobbes*, pp. 167–185.

[2] *The English Works of Thomas Hobbes*, Vol. 7, p. 186.

[3] Read before the Chicago Section of the American Mathematical Society, April, 1912. Those unfamiliar with the notation and functions employed in the theory of compound interest may consult *Text-Book of the Institute of Actuaries*, Part I, by Ralph Todhunter; *The Mathematical Theory of Investment*, by Ernest B. Skinner; *Bulletin of the Department of Agriculture*, No. 136, on Highway Bonds, by Laurence I. Hewes and James W. Glover.

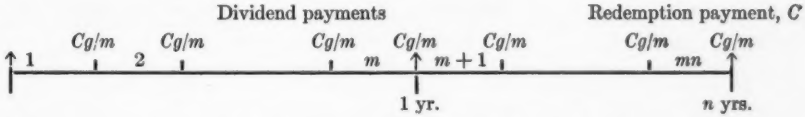
In order to derive this formula we first consider the simple case of a bond redeemed in one payment when the annual rate of dividend is g, payable in m installments, each equal to g/m, and the valuation is made at a nominal rate of interest j with frequency of conversion m.

Let C be the amount to be redeemed after n years.

Let g be the annual rate of dividend, payable in m equal installments g/m, per unit of the redemption fund C.

Let j be the nominal rate of interest, with frequency of conversion m, to be employed in the valuation.

Let A be the value of, or bid upon, the security.



Referring to the figure we see that the value of the security consists of two parts:

1. A series of dividend payments of Cg/m at the end of each mth part of the year for n years, and

2. The sum C to be redeemed at the end of n years.

The first part may be regarded as an immediate annuity-certain with payments of Cg/m per interval and running for mn intervals. Since interest is at the rate j/m per interval, the present value of this annuity is

$$a_{\overline{mn}|} \cdot Cg/m = \frac{1 - v^{mn}}{j/m} \cdot Cg/m = C(1 - v^{mn})g/j, \quad \text{where} \quad v = 1/(1 + j/m).$$

The present value of C, due in n years, or at the end of mn intervals, is Cv^{mn}, hence the present value of the security is

$$A = Cv^{mn} + (g/j)(C - Cv^{mn}), \tag{1}$$

where the function v^{mn} is to be taken at the rate j/m.

We are now prepared to consider the more general problem of valuing a security of the following nature:

1. The security is redeemed in r equal installments.

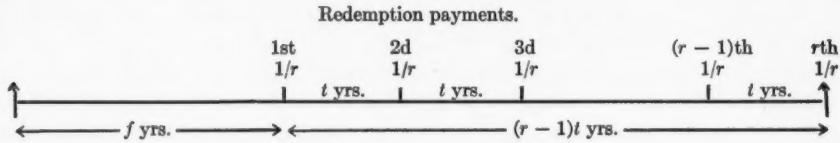
2. The first redemption payment is made at the end of f years.

3. The remaining $(r - 1)$ redemption payments are made at intervals of t years.

4. The annual rate of dividend is g and dividends are paid in equal installments at the ends of the m equal intervals into which the year is divided.

5. The security is valued at the nominal rate $j_{(m)}$.

We proceed to find the present value A of a security of the above type whose total redemption fund is unity. Since the unit fund is redeemed in r equal installments each one will be $1/r$; the following figure illustrates the nature of the security.



It is evident that the total unit loan may be regarded as made up of r separate loans, each of $1/r$, and to determine its present value we need only to find the values of these several partial loans and add them together. For this purpose we set forth the following scheme:

No. of Loan.	Present Value.	Due or Matures in Years.	Amount Maturing per Unit of Total Sum to be Redeemed.
1	A_1	$n_1 = f$	$C_1 = 1/r$
2	A_2	$n_2 = f + t$	$C_2 = 1/r$
3	A_3	$n_3 = f + 2t$	$C_3 = 1/r$
.	.	. .	.
.	.	. .	.
.	.	. .	.
r	A_r	$n_r = f + (r-1)t$	$C_r = 1/r$

Employing formula (1) for each one of these loans we have the present values as follows:

$$A_1 = \frac{1}{r} v^{mf} + (g/j)\left[\frac{1}{r} - \frac{1}{r} v^{mf}\right],$$

$$A_2 = \frac{1}{r} v^{m(f+t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r} v^{m(f+t)}\right],$$

$$A_3 = \frac{1}{r} v^{m(f+2t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r} v^{m(f+2t)}\right],$$

. . . . . .

$$A_r = \frac{1}{r} v^{m(f+\overline{r-1}t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r} v^{m(f+\overline{r-1}t)}\right],$$

all being at rate j/m.

Adding the left-hand column we have A, the present value per unit of the total sum to be redeemed. The first column on the right is a geometric series whose sum is $v^{mf}(1 - v^{mtr})/(1 - v^{mt})$, apart from the factor $1/r$, hence

$$A = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} + (g/j)\left[1 - \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right].$$

For convenience, letting

$$z = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}},$$

the preceding formula may be written

$$A = z + (g/j)(1 - z),$$

whence the *premium* $k = A - 1$ takes the form

$$k = (1 - z)(g - j)/j. \tag{2}$$

Since z is the present value of a number of quantities whose sum is unity, $1-z$ must be positive, and formula (2) shows that the premium k is positive or negative according as the rate of dividend g is greater or less than the rate of interest j desired to be realized by the investor. When k is positive the security is said to be bought at a *premium*, when negative, at a *discount*.

The formula for the premium k on the unit loan expressed in terms of the present value or v-function is:

$$k = \left[1 - \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right](g - j)/j, \qquad \text{at rate } j/m. \tag{3}$$

The most frequent case in practice is when $m = 2$. Formula (3) then becomes

$$k = \left[1 - \frac{v^{2f}}{r} \cdot \frac{1 - v^{2tr}}{1 - v^{2t}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{4}$$

These formulas may be modified somewhat since

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{v^{mf}}{r} \cdot \frac{\dfrac{1 - v^{mtr}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{v^{mf}}{r} \cdot \frac{a_{\overline{mtr}|}}{a_{\overline{mt}|}},$$

where $a_{\overline{n}|}$ is the present value of an immediate annuity-certain and one of the usually tabulated interest functions. We have then

$$k = [1 - v^{mf} a_{\overline{mtr}|}/ra_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/m, \tag{5}$$

and in the special case when $m = 2$,

$$k = [1 - v^{2f} a_{\overline{2tr}|}/ra_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/2. \tag{6}$$

It may be found desirable to express the value of the premium k in terms of the function $a_{\overline{n}|}$ and this can be accomplished as follows:

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{1}{r} \cdot \frac{v^{mf} - v^{m(f+tr)}}{1 - v^{mt}}$$

$$= \frac{1}{r} \cdot \frac{\dfrac{1 - v^{m(f+tr)}}{j/m} - \dfrac{1 - v^{mf}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{1}{r} \cdot \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{a_{\overline{mt}|}},$$

where all the annuities are to be taken at rate j/m.

This leads to the formula

$$k = \left[1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{ra_{\overline{mt}|}}\right](g - j)/j, \qquad \text{at rate } j/m, \tag{7}$$

and for the important practical case $m = 2$,

$$k = \left[1 - \frac{a_{\overline{2(f+tr)}|} - a_{\overline{2f}|}}{ra_{\overline{2t}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{8}$$

Believing that the readers of the MONTHLY may be interested in the operation of a bond loan of this character we give the following example.

What is the premium on $100,000 highway bonds, interest 5% payable semi-annually, dated January 1, 1914, maturing $50,000 January 1, 1917, and $50,000 January 1, 1919, to net the purchaser 4 per cent. compounded semiannually?

Here $f = 3$, $r = 2$, $t = 2$, $m = 2$, $g = .05$, $j = .04$, hence $m(f + tr) = 14$, $mf = 6$, $mt = 4$. Consulting a table[1] of annuities, $a_{\overline{n}|}$, with 2 per cent. as the rate of interest, and employing formula (8), the numerical work may be outlined as follows:

$a_{\overline{14}|}$ =	12.10624877	(1)
$a_{\overline{6}|}$ =	5.60143089	(2)
$a_{\overline{14}|} - a_{\overline{6}|}$ =	6.50481788	(3)
(3) ÷ 2 =	3.25240894	(4)
$a_{\overline{4}|}$ =	3.80772870	(5)
(4) ÷ (5) =	.85415984	(6)
Complement of (6) = 1 − (6) =	.14584016	(7) = first factor
(.05 − .04)/.04 =	.25	(8) = second factor
k = (7) × (8) =	.03646004	

The bid on one dollar is 1.03646004, hence the bid on the entire issue is $103,646.004. The progress of the loan is indicated in the following schedule.

(1) Interval.	(2) Year.	(3) Book Value or Principal at Beginning of Interval.	(4) Semiannual Interest of 2%.	(5) Semiannual Dividend of 2½% on Bonds.	(6) Amortization of Premium at End of Interval.	(7) Redemption Payment at End of Interval.
1	½	$103,646.00	$2,072.92	$2,500.00	$427.08	0.00
2	1	103,218.92	2,064.38	2,500.00	435.62	0.00
3	1½	102,783.30	2,055.67	2,500.00	444.33	0.00
4	2	102,338.97	2,046.78	2,500.00	453.22	0.00
5	2½	101,885.75	2,037.72	2,500.00	462.28	0.00
6	3	101,423.47	2,028.47	2,500.00	471.53	$50,000.00
7	3½	50,951.94	1,019.04	1,250.00	230.96	0.00
8	4	50,720.98	1,014.42	1,250.00	235.58	0.00
9	4½	50,485.40	1,009.71	1,250.00	240.29	0.00
10	5	50,245.11	1,004.89	1,250.00	245.11	50,000.00
Totals		817,699.84	16,354.00	20,000.00	3,646.00	100,000.00

[1] Compound Interest and Annuity Calculations with Tables, W. M. J. Werker.

By adding the several columns in this schedule various checks are obtained which are too evident to need comment. Attention is called to the fact, however, that it is not necessary to construct the whole schedule in order to determine an item in a given row and column. For example the fifth item in column three, $101,885.75, representing the present value of the outstanding security at the beginning of the fifth interval, can be calculated directly by making the proper substitutions in formula (8). When this is known the other items in the same row can be determined at once.

There are several special cases of formula (7) which deserve mention. The most common type of serial bond bears semiannual dividends and is redeemed in n equal *annual* installments, the first of which is paid at the end of the first year. In this case $f = t = 1$, $r = n$, $m = 2$, and

$$k = \left[1 - \frac{a_{\overline{2n+2}|} - a_{\overline{2}|}}{n a_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{9}$$

Since

$$a_{\overline{2n+2}|} = a_{\overline{2}|} + v^2 a_{\overline{2n}|} \quad \text{and} \quad v^2/a_{\overline{2}|} = 1/s_{\overline{2}|},$$

formula (9) may be written

$$k = \left[1 - \frac{a_{\overline{2n}|}}{n s_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{10}$$

When the serial bond is like the preceding except that dividends are payable and interest is convertible annually we have $f = t = 1$, $r = n$, $m = 1$, and

$$k = \left[1 - \frac{a_{\overline{n+1}|} - a_{\overline{1}|}}{n a_{\overline{1}|}}\right](g - j)/j, \qquad \text{at rate } j/1.$$

In this case the nominal rate $j_{(1)}$ equals the effective rate i and, since $a_{\overline{n+1}|} = a_{\overline{1}|}(1 + a_{\overline{n}|})$, the formula reduces to

$$k = \left[1 - \frac{a_{\overline{n}|}}{n}\right](g - i)/i. \tag{11}$$

When the bond is redeemed in a single installment at the end of its term, say n years, we have, $f = n$, $r = 1$, and formula (7) reduces to

$$k = \left[1 - \frac{a_{\overline{m(n+t)}|} - a_{\overline{mn}|}}{a_{\overline{mt}|}}\right](g - j)/j,$$

but since

$$a_{\overline{mn+mt}|} = a_{\overline{mn}|} + v^{mn} \cdot a_{\overline{mt}|},$$

the formula for the premium may be written

$$k = \left[\frac{1 - v^{mn}}{j/m}\right](g - j)/m, \qquad \text{at rate } j/m$$

or finally,

$$k = a_{\overline{mn}|}(g - j)/m, \qquad \text{at rate } j/m. \tag{12}$$

When, as is usually the case, dividends on the bond are paid semiannually and it is valued to net the purchaser a nominal rate j convertible twice a year, formula (12) becomes

$$k = a_{\overline{2n}|}\,(g - j)/2, \qquad \text{at rate } j/2. \tag{13}$$

In the simplest of all cases, dividends payable annually at rate g, interest compounded annually at the effective rate i, and the bond maturing in one sum at the end of n years, (12) reduces to the well-known form

$$k = a_{\overline{n}|}\,(g - i). \tag{14}$$

This formula admits of a simple interpretation because it states that the premium per unit of the sum to be redeemed is equal to the present value of an annuity whose annual rent is equal to the excess of the rate of dividend over the rate of interest desired to be realized by the purchaser. I may add that practically all the formulas in this paper admit of a direct interpretation. The interpretation of the final formula usually suggests a simple method of deriving it by general considerations and always throws a great deal of light upon the nature of the problem. It is not my purpose, however, to enter at this time upon the subject of interpretation of interest formulas.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL.

NEW QUESTIONS.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers will be printed in this column next month.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLIES.

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics? (*Note.*—An individual correspondent need not answer all the questions in number 9; it is sufficient if he answers only one.)

REPLY BY ROY CUMINS, Columbia University, N. Y.

At present there exists in the United States a decided movement toward breaking down the barriers that have hitherto kept separate the various branches

of secondary mathematics. These branches are, let us say, algebra, geometry, trigonometry, and an extension of arithmetic; but this discussion will deal primarily with algebra and geometry.

At least three words are popularly employed to designate this movement. They are fusion, unification, and correlation. Much of the difference of opinion that has been expressed with regard to the movement grows out of the fact that these words are not defined in their relation to the teaching and the subject-matter of secondary mathematics. Moreover, a person who might favor correlation opposes, say, unification, without mentioning correlation, and thus is inferentially antagonistic to all phases of the movement.

Let us inquire, then, concerning the meaning of the three words as applied to secondary mathematics. In the first place they concern not so much the content of the subject-matter of algebra and geometry as they do the arrangement and treatment of this subject-matter. This statement cannot be pushed too far, as the texts bearing the names of Myers, Cobb, Long and Brenke, and Short and Elson touch more intimately the actual out-of-school life of the pupil and find the source of more of their applications in such fields as physics, mechanics, biology, engineering, design, architecture, and the trades and industries than did the texts commonly used a decade ago. But the basis for defining our terms must chiefly be sought elsewhere.

Of the three words, "fusion" is certainly the strongest. It may mean that the subject-matter of algebra and that of geometry are to be "fused" in the way in which metals are fused to form an alloy or amalgam. Or, if we do not go quite so far as that, it means fusion to the extent to which the theorems of plane geometry are fused in the ordinary high-school text in geometry. Finally, the word may signify only the fusion which is exhibited in the ordinary high-school text in algebra. Certainly it should mean at least this much.

At the other end of the list, "correlation" may mean almost anything. No doubt we have always been correlating the subject-matter of algebra and geometry to some extent. It would be very difficult and unusual, although not impossible, to teach algebra for a year or a year and a half and then teach geometry for a year without employing in some measure the symbolism, facts, and tools of algebra.

Of course, what the advocates of the present movement wish is close and frequent correlation between the sciences of function and form. They wish coöperation and the penetration of the water-tight (may we say thought-tight?) barrier that we must admit has too effectually kept apart algebra and geometry.

Between the two terms spoken of comes "unity." To me it implies less than "fusion" and more than "correlation." As used in theme writing in my school days, it meant that everything in the subject-matter should bear on the central thought, the purpose, the organizing principle. Nothing irrelevant is to be admitted. To determine what should be admitted into a unified course in secondary mathematics, it would therefore be necessary to decide first upon an organizing principle for secondary mathematics. That task remains to be done.

Many advocates of the plan to relate more closely algebra and geometry continually cite what other countries, especially Germany and France, have done in this regard. For what is said here concerning the teaching of mathematics in these countries I am indebted to the writings of Professors Farrington, Myers, and J. W. A. Young, and to instruction by Professor C. B. Upton.

In France, says Professor Farrington, "Throughout the mathematics courses one is impressed with the intimate relations existing among the various subjects. Arithmetic is not carried to a certain point, there to give way to algebra, in its turn, perhaps, to be supplemented by geometry, but from the fifth form in one division and from the fourth form in another, at least two subjects are run conjointly." So what exists in France is not fusion or unification but parallelism of treatment and correlation. From the time algebra, geometry, and science are begun they are carried along side by side to the end of the course.

Professor Young summarizes the work in Prussia as follows: "In the German schools the subject of study is *mathematics*, and its various branches are studied side by side. At no time is only one subject being studied with the exception of the two lowest years, in which only arithmetic is taken up."

Not long ago, Professor Klein, in a conversation with Professor Upton, expressed his amusement at a seemingly prevalent idea in America that the Germans no longer teach algebra and geometry as such but fuse the two into a course in mathematics. What he says they do is to teach each of the subjects as such and let them help each other whenever they can. Clearly here again is parallelism of treatment with considerable correlation.

Professor Myers says, "The dominant phase of European secondary mathematics is of the mixed, or parallel, or quick alternation type." This does not sound like fusion or unification, as I have defined them.

What do American texts do in "fusing" algebra and geometry? For the most part they do not fuse the two at all, not even the newest texts. Their common plan is to give a chapter or two on algebra, then devote the next pages to geometry, with little or no reference to the algebra that preceded; and then follow with another chapter on algebra that is almost entirely unrelated to the geometry that preceded. (I have not considered the Myers texts in this regard, as they are now being revised, nor the Cobb text, as it is not to my knowledge extensively used in high schools.)

Why should we give parallel courses in geometry and algebra instead of the traditional tandem courses? As it is, one class contends that algebra should precede, while another asks that geometry be taught first. To a high school freshman the beginnings of geometry are certainly more simple, concrete, and familiar than those of algebra. They relate to things that the pupil actually sees out in the world or can construct for himself. On the other hand, the easier parts of algebra are far simpler than the more difficult parts of geometry. If the two were taught simultaneously, each could be of great assistance to the other and correlation could be much more easily effected than is now done. In the third place, the pupil would not rush at high pressure over each subject as he

now does. A boy in a German school, for example, studies algebra for six years and geometry for eight years. Time for the maturing of thought and for the effects of retroactive inhibition are thus afforded. Pedagogically, then, we should have parallelism of treatment.

Furthermore, as Professor Young has pointed out, we are not compelled to conduct an experiment to demonstrate the practicability of our theory. France and Germany have done that for us and they tell us with assurance that a great gain has been accomplished. What I favor, therefore, is that both algebra and geometry be begun early, with geometry first, that the study of each be extended over a far greater time than is done at present, and that we do our best in bringing the subject-matter, symbolism, and methods of each subject to assist the other. This should be called parallelism with all possible correlation.

19. How many known proofs are there of the proposition that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides? Where are the proofs to be found?

Reply by C. E. Horne, Westminster College, Colorado.

As a possible reply to the above question I should like to submit the following: Jury Wipper, *Sechsundvierzig Beweise des Pythagoräischen Lehrsatzes*. Aus dem Russischen von F. Graap, Leipzig, 1880.

If there is anything more complete than this work, I shall be glad to learn of it.

(*Note.*—It may be worth while to mention also, in this connection, the earlier and less complete work of Ignaz Hoffman, *Der Pythagoräischen Lehrsatz mit 32 Beweisen*, Mainz, 1819.—Editor.)

Reply by H. E. Slaught, The University of Chicago.

It may be of interest that the first class in geometry which I taught, in 1883, became "Pythagorean crazy," and that we elaborated over forty proofs, each being carefully copied on stiff cardboard and all bound together and deposited in the archives of the school, Peddie Institute, Heightstown, N. J. I hope they are still preserved.

22. What can the colleges do toward improving the teaching of mathematics in secondary schools?

Reply by Charles N. Moore, University of Cincinnati.

It is evidently desirable that colleges and universities should do everything they reasonably can to improve the teaching of preparatory subjects in the secondary and primary schools in their localities. Not only is this their duty from the broad viewpoint of the relation of such institutions to society; but, from the narrower viewpoint of benefits to the institutions themselves, any results achieved in this direction will react in a stimulating way upon the college or university. Consequently it seems worth while to give here a brief account of what the department of mathematics at the University of Cincinnati is and has been doing in that direction for the past fifteen years.

The University of Cincinnati, being a municipal institution, has always made a special effort to coöperate as much as possible with all municipal activities,

and particularly with all activities in other subdivisions of the public educational system of the city. In line with this general policy, courses especially designed for those who are actively engaged in teaching have been given by all the departments in the late afternoon hours and on Saturdays. These courses have been highly appreciated and well attended by teachers in the city and in the neighboring towns.

In the department of mathematics at least one such course, and sometimes three or more, have been given each year during the past fifteen years. These courses have ranged from freshman college courses to advanced graduate courses, in order to meet the needs and interests of as many teachers in the vicinity as possible. Very few of them were definitely method courses, for it is a conviction of the department, based on its own experiences, that most teachers of primary or secondary subjects who have had some experience in actual teaching will profit more from additional information in their subject than they will from specific training in methods. Of course incidental discussion of pedagogical questions is valuable, and we have always kept that in mind. But our principal effort has been to give the teachers in the preparatory schools of our vicinity a wider mathematical horizon.

As a result of our efforts in this direction almost all the younger teachers of mathematics in the Cincinnati high schools have taken one or more courses in mathematics at the University, and quite a number of them have taken the master's degree in mathematics. Some of these latter did all their mathematical study from freshman college work through to the master's degree in connection with these courses given especially for teachers. Also quite a number of teachers in the primary schools, including several who are now principals of schools in the city, and a number of teachers in preparatory schools of neighboring towns, have taken one or more courses in mathematics.

Another feature of the work done at the University of Cincinnati to bring about closer coöperation with the teachers in the preparatory schools is an annual gathering at the University of teachers in the accredited secondary schools. Those in attendance are grouped into sections according to the subjects in which their chief interest lies, and pedagogical questions relating to secondary schools are discussed. The members of the different departments of the university make it a point to attend the sectional meetings in which their subjects are discussed, and not only participate in the discussion but frequently give one of the papers of the day. The papers presented in the mathematical section and the ensuing discussions have been found to be extremely interesting and helpful to all those attending, and there is no doubt that these annual meetings have done much to increase the feeling of a community of interest between teachers in the university and those in the secondary field.

The chief apparent effect, as noticed by the department of mathematics, of this steady effort to do something toward the further training of teachers in the preparatory schools has been to unify the teaching of mathematics in this vicinity and to bring about a closer coöperation among all those engaged in such teaching,

whether the subject taught be arithmetic or the theory of functions. We are thoroughly convinced that the value of this unification and coöperation makes the effort put forth to obtain it seem very slight in comparison. We are also convinced that there are other beneficial effects of this effort which are less apparent but not less important. And, finally, we believe that any college or university that has done little or no work of this sort will find it highly worth while to increase its activity in this direction.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Analytic Geometry. By L. WAYLAND DOWLING and F. E. TURNEAURE. Henry Holt and Co., New York, 1914. xi + 266 pages.

A chief feature of several recent texts on analytics is the emphasis of the general idea of a function and its graph rather than of the theory of conics. This text retains that emphasis to a large extent. After 25 pages devoted to chapters on "Systems of Coördinates" (I) and "Directed Segments and Areas of Plane Figures" (II), we find 65 pages of discussion of "Functions and their Graphic Representation" (III), "Loci and their Equations" (IV), and "Equations and their Loci" (V), the last chapter including "Transformation of Coördinates." In Chapter III first methods of graphing functions are given with illustrations from algebraic and transcendental functions in rectangular and polar coördinates. The equation of a locus is defined in Chapter IV and the standard equations of straight lines, the conics and Cassinian ovals are derived for both coördinate systems. Chapter V gives methods of discussing an equation with numerous examples. It is not until Chapter VI, "Loci of First Order," beginning on page 98, that we find a systematic treatment of the straight line. After what has preceded, 10 pages suffices. Then in Chapter VII, "Loci of Second Order, Equations in Standard Form," we find, compassed in 34 pages, a fairly complete elementary treatment of the conics, including "Poles and Polars" and "Systems of Conics." The next chapter treats the general equation of second degree.

In accord with another modern development, we find (included in Chapter IX, "Loci of Higher Order and Other Loci") a twelve page discussion of "Empirical Equations and their Loci." This subject, on account of its importance in applications of mathematics to the sciences, seems destined to become an essential part of a good course in analytics. The presentation of the authors, which includes the use of logarithmic coördinate paper, is excellent.

Following these chapters on plane analytics is a brief treatment (about 50 pages) of solid analytics. There is little of novelty in this part of the book.

The book as a whole impresses one very favorably. The general order of presentation is excellently adapted to give the student a real appreciation of the power and beauty of analytic geometry, and also the ability to use it. The tendency to lay a little more stress than usual on the geometrical aspect of the

subject, especially in the chapter on "Loci and their Equations," by numerous figures and by various methods of constructing some of the loci, will be welcomed by many teachers.

There are several matters of detail, however, which one may criticise. Exceptional cases of various sorts are, for example, quite generally ignored. Thus in discussing the two-point form of the equation of a straight line, the exceptional cases of lines parallel to either coördinate axis receive no mention. In giving the slope forms, no mention is made of lines parallel to the y-axis. And all lines through the origin are ignored in connection with the intercept and normal forms. The determinant form of the equation of a plane through three points is said to be linear without mentioning the exceptional case arising when the points are collinear. Likewise no account is taken of exceptional cases in the discussion of pencils of lines or of conics.

While the general idea of Chapter IV, "Loci and their Equations," appeals to me, the logic seems unsatisfactory in two respects. By definition, "The equation of the locus of a point is an equation in the variables x and y which is satisfied by the coördinates of every point on the locus; and conversely. . . ." The "*and conversely*" is subsequently neglected without comment in deriving equations except in the case of the circle. This, of course, invalidates the proof, for instance, that the locus of every equation of first degree is a straight line. The proof of the "*and conversely*" for a straight line is as difficult as the direct, and the omission seems hardly excusable. The second criticism is on the expression, "*the* equation of the locus . . ." without comment on the first "*the*." There is an infinite number of equations satisfying the prescribed conditions in general, and some remark is apparently necessary before one of them may be designated as "*the* equation." For example, the following are all equations of the same real locus:

$$x^2 + y^2 = 4;\ x^4 + 2x^2y^2 + y^4 = 16;$$

$$9x^2 + 9y^2 = 36;\ x^4 + x^2y^2 + 4y^4 = 16.$$

At least a footnote of explanation seems desirable.

It is further noted that the proof of the formula for the distance from a line to a point (p. 104) is invalid for some cases (e. g., for Fig. 66, p. 105).

As a minor point for criticism, one is surprised to find that the formula for the distance between two points is not given for rectangular cartesian coördinates as distinct from oblique coördinates. Also no reason for deriving formulas for the mid-point of a line segment by a method different from that required to find the point of division in a given ratio, r, is apparent since the former is a simple corollary of the latter. And although the latter is given in the usual manner, it is open to a certain criticism. A point P is found on P_1P_2 such that $P_1P/PP_2 = r$; its coördinates are

$$x = \frac{x_1 + rx_2}{1 + r};\ y = \frac{y_1 + ry_2}{1 + r}.$$

There is such a point for all values of r *except* $r = -1$; and by taking all values of r we get all points on the line P_1P_2 *except* P_2. These exceptional cases are avoided and we get simpler formulas, which are more easily derived, if we determine P such that $P_1P/P_1P_2 = r$ as follows; namely[1]

$$x = x_1 + r\cdot\Delta x = x_1 + r(x_2 - x_1), \qquad y = y_1 + r\cdot\Delta y = y_1 + r(y_2 - y_1).$$

To sum up, although marred by several inaccuracies in detail, the book as a whole is very good, thoroughly modern, and includes much in a small compass.

E. J. MOULTON.

NORTHWESTERN UNIVERSITY.

The Algebra of Logic. By LOUIS COUTURAT. Authorized translation by LYDIA G. ROBINSON, with a preface by PHILIP E. B. JOURDAIN. The Open Court Publishing Co., Chicago, 1914. xiv + 98 pages. $1.50.

This volume is a translation of volume number 24 in the Gauthier-Villars collection under the general title Scientia. These very useful manuals run about 100 pages each, in two series, the present number belonging to the physico-mathematical series. The translation has been well done although it is not exactly a literal rendering of the French style of sentence. A bibliography has been inserted and the preface presents, in brief form, a good history of the development of symbolic logic. These additions are distinct improvements and will make the book more useful.

The algebra of logic, as its name indicates, is a mathematical treatment of logic. It is not a new species of numerical calculation by symbols, but is a treatment of the field of logical notions by introducing symbols which have to be combined according to definite laws. It is an application of the methods of mathematics to logic, and is just as much applied mathematics as is mechanics. The only peculiarity in the situation is that in the reasoning processes employed, the symbols and laws relate to the reasoning process itself. The field of study in logic consists of classes of objects of the mind as marked by some distinguishing quality and the relations between such classes; assertions regarding mental objects and their relations; and relations themselves as objects with relations to each other. For example we may study the class of triangles that are distinguished as isosceles, and the class that are distinguished as having two equal angles. The identification of the two classes is done by logical processes. If we consider the problem of identification of classes of objects in general we are studying logic. Just as no one can get along in the world without using arithmetic consciously or unconsciously, so no one can get along without reasoning. If he becomes interested in the processes of arithmetic and the study of identities between numerical expressions in general, then he begins to study algebra. So too if he becomes interested in the process of reasoning he begins to study logic, and the most efficient way to do this is to apply the mathematical method; that is to say, to devise a system of symbols that will take the place of the mental objects under study, to parallel the mental combinations of these objects with

[1] These formulas are given, for example, by Ziwet and Hopkins, *Analytic Geometry*, p. 8.

combinations of the symbols, and to state as laws of operation the results of the mental transition from object to object.

Consider, for example, the process of reasoning called the syllogism. In its most simple form this consists in the succession of statements:

If objects A defined in any way have the quality B, and if objects that have the quality B must have the quality C, then the objects called A must have the quality C. We state this symbolically

$$(A < B)(B < C) < (A < C).$$

We are in a much better position to discuss the syllogism from this visualization of it in symbols than if we try to discuss it in words and in sentences. We can indeed reduce the manipulation of our logical formulæ to a few very simple rules which not only enable us to produce the usual results of logic, but we can solve very much more difficult questions. For examples see Jevons' *Principles of Science*, or his *Studies in Deductive Logic*.

We can do more however, for if A, B, C above are not simple classes but are propositions, we have a formula relating them in precisely the same manner as the syllogism above, and all our formulæ become susceptible of another interpretation. We can thus reduce our two-fold logic in the main to a single treatment, which is a distinct gain. The author does this throughout the book. The formulæ developed are interesting in showing what can be done in the discussion of general questions. However it should be remembered that for any concrete case, the fundamental rules of the symbolism are all one needs to solve the problem.

A word is in place perhaps as to why one should study symbolic logic at all. The best answer is that it is desirable for precisely the reasons that one should study algebra, or theory of groups. While everyone possibly is gifted with correct reasoning power, at least so long as the problem is simple, it is often not possible to carry an intricate set of premises in the mind and deduce the correct conclusion. Nor is it possible easily to solve the reverse cases and say, for example, what would be the simplest set of hypotheses that would make certain propositions follow as consequences. With the machinery of symbolic logic this becomes quite manageable.

As an example we mention the famous problem of Venn stated thus:

The members of a board are either bondholders or shareholders, but not both; all the bondholders are on the board. What conclusions can be drawn?

As an example of inverse logical problems we may quote the following: What are the premises that enable us to conclude that objects with the quality C do not have the quality E, and that objects without the quality C must have the quality E, do not have the quality A, and also either do not have the quality B or else have the quality B but do not have the quality D?

These simple problems illustrate the character of work that can be handled easily by symbolic logic but not easily in the ordinary way. They exemplify in their solution also the fact that the syllogism is only one of many ways of

reasoning. This however we have not space to enter into, and it should be followed up after reading the book before us, in some of the more extended treatises mentioned in the bibliography in this book. Every student of mathematics ought to read through some book on logic, and the present one is an admirable introduction to the subject.

JAMES BYRNIE SHAW.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

Note.—An additional supply of good live problems is desired, especially in algebra, calculus, and mechanics.—EDITORS.

ALGEBRA.

When this issue was made up, solutions had been received for numbers 424–427

428. Proposed by FRANK IRWIN, University of California.

If the roots of the equation

$$x^n - na_1x^{n-1} + \binom{n}{2} a_2x^{n-2} + \cdots = 0$$

are all real, the condition that they should all be equal is $a_1^2 = a_2$. A proof of the sufficiency of the condition is readily obtained from a consideration of derivatives. A proof is desired not based on such considerations.

429. Proposed by C. N. SCHMALL, New York City.

It is given that d_1, d_2, d_3, are the greatest common divisors of y and z, z and x, x and y, respectively; also that m_1, m_2, m_3, are the least common multiples of the same pairs of numbers. If d and m are the greatest common divisor and least common multiple, respectively, of x, y, and z, show that

$$\frac{m}{d} = \left(\frac{m_1m_2m_3}{d_1d_2d_3}\right)^{1/2}.$$

430. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the equations algebraically and also graphically: $x^y + y^x = xy$, $\quad x^x + y^y = x + y$.

GEOMETRY.

When this issue was made up, solutions had been received for numbers 452–454

457. Proposed by NATHAN ALTSHILLER, University of Washington.

AB and AC are respectively a diameter and a chord of a circle whose center is O. The lines joining B to the extremities of the diameter perpendicular to AC, meet AC in the points M, N. Express the angle MON in terms of the angle CAB.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Given edges l, m, and n of a parallelopiped and the angles a, b, and c which the edges make with one another. Show that, if $s = (a + b + c)/2$, the volume equals

$$2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

459. Proposed by C. N. SCHMALL, New York City.

In a right triangle ABC, right-angled at C, a point F is taken in the side CB and perpendiculars CD and FE are dropped on the hypothenuse AB. Prove $AD \cdot AE + CD \cdot EF = \overline{AC}^2$.

CALCULUS.

When this issue was made up, solutions had been received for numbers 366–377

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y = x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A = 2\pi \int_0^1 x^n \sqrt{1 + n^2 x^{2n-2}}\, dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n=\infty} \int_0^1 x^n \sqrt{1 + n^2 x^{2n-2}}\, dx = \tfrac{1}{2}.$$

379. Proposed by C. N. SCHMALL, New York City.

Express the equation of the folium, $x^3 + y^3 = 3axy$, in parametric form and find the area of the loop.

(From E. B. Wilson's *Advanced Calculus*, p. 296, ex. 5.)

MECHANICS.

When this issue was made up, solutions had been received for numbers 297, 301, and 302

303. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A pile-driver weighing 500 pounds falls through 10 ft. and drives a pile weighing 400 pounds 3 inches into the ground. Show that the average force of the blow is $11{,}111\frac{1}{9}$ pounds.

NUMBER THEORY.

When this issue was made up, solutions had been received for numbers 224, 225, 226, and 229

228. Proposed by HERMON C. KATANIK, Indianapolis, Ind.

Deduce a formula for the difference between any two squares, and thus show that (1) The difference between any two consecutive squares is of the form $2p + 1$; (2) The difference between any two squares is even or odd according to whether they are separated by an odd or even number of squares; (3) The differences of the squares of the consecutive terms of any arithmetic progression form another arithmetic progression.

229. Proposed by WALTER C. EELLS, U. S. Naval Academy.

If p and q are integers and p is prime and positive, find the condition on q that the equation $p^x = qx$ shall have integral solutions, solve for x, and show that for a special value of p it has two solutions for a given q, otherwise only one.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

418. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Form the algebraic equation whose roots are

$$a_1 = 2\cos\left(\frac{2\pi}{15}\right), \quad a_2 = 2\cos\left(\frac{4\pi}{15}\right), \quad a_3 = 2\cos\left(\frac{8\pi}{15}\right), \quad a_4 = 2\cos\left(\frac{14\pi}{15}\right).$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The equation, $x^{15} - 1 = 0$, has for its roots

$$r_k = \cos\left(\frac{2k\pi}{15}\right) + i \sin\left(\frac{2k\pi}{15}\right),$$

where $k = 1, 2, 3, \cdots, 15$; $r_k + \frac{1}{r_k} = 2\cos\left(\frac{2\pi k}{15}\right)$. If, then, we divide out all the roots that are not primitive, leaving only $r_1, r_2, r_4, r_7, r_8, r_{11}, r_{13}, r_{14}$, and, noting that $r_{14} = \frac{1}{r_1}$, $r_{13} = \frac{1}{r_2}$, $r_{11} = \frac{1}{r_4}$, $r_8 = \frac{1}{r_7}$, then reduce this equation by the usual method for a reciprocal equation, we shall have the desired equation. This gives us $x^4 - x^3 - 4x^2 + 4x + 1 = 0$.

In a similar manner we can form the equation whose roots are all distinct values of $2\cos\left(\frac{2\pi k}{n}\right)$, where k takes on all integral values, prime to n between and including 1 and $n - 1$.

Also solved by A. M. HARDING and the PROPOSER.

419. Proposed by GEORGE A. OSBORNE, Massachusetts Institute of Technology.

Show that

$$1^5 + 2^5 + 3^5 + \cdots + n^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 = 2(1 + 2 + 3 + \cdots + n)^4.$$

SOLUTION BY ELMER SCHUYLER, Brooklyn, N. Y.

By induction we must show that

$$(A) \qquad \begin{aligned} &1^5 + 2^5 + 3^5 + \cdots + n^5 + (n+1)^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 \\ &\qquad + (n+1)^7 = 2(1 + 2 + 3 + \cdots + n + (n+1))^4 \end{aligned}$$

Starting with the right member of (A), we have

$$\begin{aligned} &2(1 + 2 + 3 + \cdots + n + \overline{n+1})^4 \\ &\quad = 2\left[\left\{\frac{n(n+1)}{2}\right\}^4 + 4\left\{\frac{n(n+1)}{2}\right\}^3 (n+1) + 6\left\{\frac{n(n+1)}{2}\right\}^2 (n+1)^2 \right. \\ &\qquad \left. + 4\left\{\frac{n(n+1)}{2}\right\}(n+1)^3 + (n+1)^4\right] \\ &\quad = 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^4(n^3 + 3n^2 + 4n + 2) \\ &\quad = 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^5(n^2 + 2n + 2) \\ &\quad = 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^5 + (n+1)^7, \end{aligned}$$

which, if the given equation holds, reduces to

$$1^5 + 2^5 + 3^5 + \cdots n^5 + (n+1)^5 + 1^7 + 2^7 + 3^7 + \cdots n^7 + (n+1)^7,$$

and this is the left member of (A). Hence the induction is established.

Also solved by GEORGE Y. SOSNOW, WALTER C. EELLS, J. L. RILEY, CLIFFORD N. MILLS, JAMES A. BULLARD, ELBERT H. CLARKE, A. M. HARDING, R. M. MATHEWS, B. F. YANNEY, ELIJAH SWIFT, and S. A. JOFFE.

GEOMETRY.

433. Proposed by R. P. BAKER, University of Iowa.

A transformation of the plane keeping the radius of curvature of all curves invariant is either (1) a real or imaginary motion or reflexion, or (2) not a point transformation.

Note.—By mistake this problem was originally credited to W. H. BUSSEY, instead of to R. P. BAKER, who is the rightful Proposer.—EDITORS.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Suppose that this is a point transformation transforming points in plane I to points in plane II. If one configuration is transformed into another, the second is called the *image* of the first. The theorem is not necessarily true if only a finite number of points or even a one-dimensional set be transformed, and I shall assume that the transformation is defined over a two-dimensional region of I possessing at least one interior point and over a similar region of II, and that it is one-to-one in these regions. Now the image of a straight line is a straight line and that of a circle an equal circle. Hence the transformation satisfies the conditions given in an article of mine in the *Bulletin of the American Mathematical Society* (2d series, vol. x, 1903–1904, pp. 247 ff.), and hence must be a collineation. If it is to transform a circle into a circle, it must leave the circular points at infinity invariant (or interchange them) and hence is a finite collineation of the form $x' = a_1x + b_1y + c_1$, $y' = a_2x + b_2y + c_2$. Omitting the c's, which corresponds to making a translation, and writing the condition that $x^2 + y^2 = r^2$ is carried into itself, we obtain the equations $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$, $a_1b_1 + a_2b_2 = 0$. If we put $a_1 = \cos\alpha$, $a_2 = \sin\alpha$, then we find that $b_1 = \sin\alpha$, $b_2 = -\cos\alpha$, or $b_1 = -\sin\alpha$, $b_2 = \cos\alpha$. Our collineation is then either of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= \sin\alpha x - \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to (1) a rotation through the angle α, (2) a translation through a distance c_1 along the x-axis, and a distance $-c_2$ long the y-axis, and (3) a reflexion in the x-axis; or else of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= -\sin\alpha x + \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to a rotation followed by a translation.

447. Proposed by HORACE OLSON, Chicago, Ill.

Given the edge of a regular tetrahedron, find the radius of the circumscribed sphere.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory.

Let A-BCD be a regular tetrahedron, whose edge is a. Let the radius of the circumsphere be R. In the $\triangle BCD$, let the medians meet in O. Then $BC = a$; $BB' = \frac{a\sqrt{3}}{2}$; $BO = \frac{2}{3} \times \frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{3}$. Since $\triangle BCD$ is equilateral, O is

equidistant from BC, CD, and BD, and hence equidistant from the planes A-BC, A-CD, and A-BD. Hence O lies on the line of intersection of the three planes bisecting the dihedral angles C-AB-D, B-AD-C, and D-AC-B.

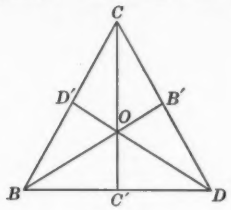


FIG. 1.

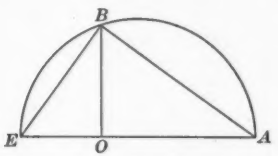


FIG. 2.

But these three planes intersect in the diameter through A of the circumscribed sphere. Extend AO to meet the sphere again in E (Fig. 2). Then EA = diameter of circumsphere = $2R$. Since A-BCD is a regular tetrahedron, AO is perpendicular to the base, BCD. Hence $\angle AOB$ is a right angle. Also $\angle EBA$, being inscribed in a semicircle, is a right angle. Hence in right $\triangle BOA$,

$$\overline{OA}^2 = \overline{AB}^2 - \overline{BO}^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3},\ OA = \frac{a\sqrt{2}}{\sqrt{3}}.$$

In right $\triangle EBA$, $\overline{BA}^2 = EA$. $OA = 2R \cdot \dfrac{a\sqrt{2}}{\sqrt{3}}$.

$$a^2 = 2R \cdot \frac{a\sqrt{2}}{\sqrt{3}}, \qquad R = \frac{a\sqrt{3}}{2\sqrt{2}} = \frac{a\sqrt{6}}{4}.$$

Also solved by R. M. MATHEWS, NATHAN ALTSHILLER, A. M. HARDING, CLIFFORD N. MILLS, WALTER C. EELLS, A. H. HOLMES, HORACE OLSON, J. C. CLAGETT, J. W. CLAWSON.

CALCULUS.

361. Proposed by EMMA M. GIBSON, Drury College.

Determine the system of curves satisfying the differential equation

$$[(1 + x^2)^{1/2} + ny]dx + [(1 + y^2)^{1/2} + nx]dy = 0,$$

and show that the curve which passes through the point $(0, n)$ contains as part of itself the conic

$$x^2 + y^2 + 2xy(1 + n^2)^{1/2} = n^2.$$

(From Forsyth's *Differential Equations*, p. 41.)

SOLUTION BY GEO. W. HARTWELL, Hamline University.

The terms of the given differential equation may be arranged as follows:

$$(1 + x^2)^{\frac{1}{2}}dx + (1 + y^2)^{\frac{1}{2}}dy + nxdy + nydx = 0 \qquad (1)$$

and the equation integrated immediately, giving

$$x\sqrt{1 + x^2} + y\sqrt{1 + y^2} + 2nxy + \log\,(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = c. \qquad (2)$$

The equation of the curve of this system passing through $(0, n)$ is then

$$x\sqrt{1+x^2}+y\sqrt{1+y^2}-n\sqrt{1+n^2}+2nxy + \log\left[\frac{(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})}{n+\sqrt{1+n^2}}\right]=0. \tag{3}$$

Solving the equation of the conic for x we have

$$x=-y\sqrt{1+n^2}\pm n\sqrt{1+y^2}.$$

These values of x satisfy (3); hence, the conic must be a part of the curve represented by (3).

362. Proposed by C. N. SCHMALL, New York City.

Having given $y^3-a^2y+axy-x^3=0$, show by Maclaurin's theorem that

$$y=-\frac{x^3}{a^2}-\frac{x^4}{a^3}-\frac{x^5}{a^4}-\cdots,$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We obtain by successive differentiation

$$(3y^2-a^2+ax)\frac{dy}{dx}-3x^2+ay=0,$$

$$(3y^2-a^2+ax)\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2+2a\frac{dy}{dx}-6x=0,$$

$$(3y^2-a^2+ax)\frac{d^3y}{dx^3}+18y\frac{d^2y}{dx^2}\frac{dy}{dx}+6\left(\frac{dy}{dx}\right)^3+3a\frac{d^2y}{dx^2}-6=0,$$

$$(3y^2-a^2+ax)\frac{d^4y}{dx^4}+24y\frac{d^3y}{dx^3}\frac{dy}{dx}+18y\left(\frac{d^2y}{dx^2}\right)^2+36\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2+4a\frac{d^3y}{dx^3}=0,$$

$$(3y^2-a^2+ax)\frac{d^5y}{dx^5}+30y\frac{d^4y}{dx^4}\frac{dy}{dx}+60y\frac{d^3y}{dx^3}\frac{d^2y}{dx^2}+60\frac{d^3y}{dx^3}\left(\frac{dy}{dx}\right)^2 + 90\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}+5a\frac{d^4y}{dx^4}=0.$$

When $x=0$, $y=0$, a, or $-a$. Choosing the first value of y, we obtain

$$\frac{dy}{dx}=0,\quad \frac{d^2y}{dx^2}=0,\quad \frac{d^3y}{dx^3}=-\frac{6}{a^2},\quad \frac{d^4y}{dx^4}=-\frac{24}{a^3},\quad \frac{d^5y}{dx^5}=-\frac{120}{a^4}.$$

Hence,

$$y=-\frac{6}{a^2}\frac{x^3}{3!}-\frac{24}{a^3}\frac{x^4}{4!}-\frac{120}{a^4}\frac{x^5}{5!}-\cdots$$

$$=-\frac{x^3}{a^2}-\frac{x^4}{a^3}-\frac{x^5}{a^4}-\cdots.$$

Also solved by PAUL CAPRON and GEO. W. HARTWELL.

NUMBER THEORY.

215. Proposed by R. D. CARMICHAEL, Indiana University.

Find one or more values of n such that a polygon of n sides shall have the number of its diagonals equal to the cube of an integer.

SOLUTION BY WALTER C. EELLS, U. S. Naval Academy.

Since the number of diagonals of an n-sided polygon is $\frac{n(n-3)}{2}$, this is equivalent to the problem: find solutions in integers of $n^2 - 3n = 2k^3$. By the aid of a table of squares it is easily found that the only values of n less than 1,000 satisfying this equation are 9 and 128, for which $k = 3$ and 20 respectively, the number of diagonals being 27 and 8,000.

216. Proposed by ELIJAH SWIFT, University of Vermont.

If p is a prime > 3, show that $\Sigma_{a=1}^{a=p-1} 1/a \equiv 0 \pmod{p^2}$, where $1/a \equiv x$, if $ax \equiv 1 \pmod{p^2}$.

SOLUTION BY THE PROPOSER.

Referring to my solution[1] of algebra problem number 385, I proved that

$A_{p-2} \equiv 0 \pmod{p^2}$, where $A_{p-2} \equiv 1\cdot2\cdot3\cdots p-2 + 1\cdot3\cdot4\cdots p-1 + \cdots$.

Hence,

$$A_{p-2} \equiv (1\cdot2\cdot3\cdots(p-1))\sum_{a=1}^{p-1}\frac{1}{a}.$$

Suppose that

$$1\cdot2\cdot3\cdots(p-1) \equiv -1 + A\cdot p \pmod{p^2}.$$

Then

$$A_{p-2} \equiv \sum_{a=1}^{p-1}\frac{-1+Ap}{a} = \sum_{a=1}^{p-1}\frac{-1}{a} + pA\sum_{a=1}^{p-1}\frac{1}{a}.$$

But

$$\sum_{a=1}^{a=p-1}\frac{1}{a} \equiv \sum_{a=1}^{a=p-1} a \pmod{p} \equiv 0,$$

since

$$A_{p-2} \equiv 0 \pmod{p^2} \quad \sum_{a=1}^{a=p-1}\frac{1}{a} \equiv 0 \pmod{p^2}.$$

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

Miss MARIE GUGEL, formerly teacher of mathematics in the Toledo, Ohio, high school, is now supervisor of high schools in Columbus. She is secretary of the mathematics section of the Central Association of Science and Mathematics Teachers.

Mr. FORREST R. BAKER, assistant in mathematics at the University of Michigan, died December 6, following an operation for appendicitis.

[1] Volume XXI, page 157, May, 1914.

Professor L. C. KARPINSKI has been promoted to a junior professorship of mathematics in the University of Michigan.

Mr. F. A. FORAKER, of the University of Pittsburgh, contributes to *Education* for December, 1914, an article on "The relation of the symbols of mathematics to the elements of the problems."

At the University of Oklahoma Associate Professor F. C. KENT has resigned and Mr. H. B. GOSSARD, of the Johns Hopkins University, has been appointed to an instructorship in mathematics.

The *Bollettino di bibliografia e storia delle scienza matematiche* for the last quarter of 1914 contains an Italian translation of Professor E. J. WILCZYNSKI'S paper on "Some general aspects of modern geometry" which appeared in the *Bulletin of the American Mathematical Society* for April, 1913. The translation is preceded by a brief appreciative account of Professor Wilczynski's fundamental work on projective differential geometry.

An article in the October, 1914, number of the *Bulletin* of the Society for the Promotion of Engineering Education, entitled "What a technical education costs," gives the average annual expense of 65 students at the Massachusetts Institute of Technology. This amounted to $616.39 for students from a distance and $327.65 for students living at home.

School Science and Mathematics for November and December, 1914, printed a paper on "Some observations on the study and teaching of mathematics in Germany," read by Professor GORDON N. ARMSTRONG, of Ohio Wesleyan University, before the April meeting of the Ohio Teachers of Mathematics and Science.

In the *Mathematical Gazette* for October, 1914, Mr. W. H. MACAULAY presents the results of an investigation of the problem of dissecting two given rectilineal figures of equal areas, by straight lines, so that the parts of either will fit on the other. The simplest case is that in which the triangles ABC and $A'B'C'$ have a pair of equal sides, AB and $A'B'$, and do not differ too much in shape. They can be divided by a three-part and by a four-part dissection, the former being accomplished by joining E and D, the mid points of AC and BC, to F in AB by lines equal to half of $B'C'$ and $A'C'$ respectively; and analogous lines in triangle $A'B'C'$. With the aid of these two fundamental cases Mr. Macaulay establishes other cases, such as the dissection of any given pair of triangles of equal areas (not differing too much in shape) by a four-part and by a seven-part dissection; of a rectangle and square of equal areas; of a quadrilateral and parallelogram of equal areas, etc.

A recent letter from PROFESSOR CAJORI includes the following interesting notes:

A "Junior Encyclopedia Britannica" is now in preparation. The mathematical articles will be prepared by PHILIP E. B. JOURDAIN, of Girton, near Cambridge. Only the first two volumes will appear until the war is over, but these will contain the two long articles "Arithmetic" and "Algebra," as well as several shorter articles. Jourdain's idea is to treat every mathematical subject historically. For example, instead of attempting to define algebra, which word has meant different things at different times, the plan is to show that algebra grew out of such and such problems and took on such and such meanings, and to give an idea also of the modern, advanced work.

Mr. Jourdain has been working for some time, on a "History of Mathematical Thought," to be published by George Bell, of London. It will contain a very thorough treatment of the development of the leading conceptions of mathematics, such as limit, continuity, etc. The book will carry the subjects down to modern times and will not go into great details about purely technical advances. Somewhat new will be the stress laid on the influence of Zeno on the form which Greek mathematical thought took, also the unconscious and illogical way in which negative, irrational and imaginary numbers were introduced. A good deal of space will be devoted to the modern work and the principles of mathematics.

The Open Court Publishing Co. will shortly bring out two books translated and annotated by Mr. Jourdain. One is a supplementary volume to the English Mach's *Mechanics*, giving the additions made by Mach to the latest German edition. The other book is a translation of Georg Cantor's papers on transfinite numbers in volumes XLVI and XLIX of the *Mathematische Annalen*.

Periodico di Matematica per l'insegnamento secondario, edited by Professor Giulio Lazzeri, is published bi-monthly at Livorno, Italy, and the *Supplemento al Periodico di Matematica*, under the same editorship, appears monthly from November to July inclusive. The former journal is now in the thirtieth year of publication, and the *Supplemento* in the eighteenth year.

The *Periodico* is intended to meet the scholarly demands of the teachers of mathematics in the secondary schools, and also the needs of university students of mathematics and physics. The *Supplemento* is more elementary in character, and aims to excite even in students in the secondary schools a love for the study of mathematics and physics. A somewhat unique feature of the latter journal is a continuous series of prize problems, the competitors being any students of mathematics and physics in the secondary schools, and the prizes being books on mathematics.

The November, 1914, issue (48 pages) of the *Periodico* contains the following articles, all in Italian: C. MINEO, "On the concept of a real number and upon an elementary theorem concerning such numbers"; D. KRYJANOVSKY, "Upon maxima and minima of plane figures (continuation)"; G. LAZZERI, "Static moments, moments of inertia, and moments of higher order (continuation)"; E. PICCIOLI, "The second hypersphere of Lemoine, etc."; G. CANDIDO, "The

solution of the equation

$$\sqrt[2k+1]{A} + \sqrt[2k+1]{B} + \sqrt{C} = 0,"$$

and "On the equation, $\sum_1^k x_i^2 = y^p$". There also appears a review (less than a page) of the new 5-place logarithmic tables, 56 pages, by Professor E. MOUZIN, published by the Fratelli Bocca in Rome.

The November, 1914, issue of the *Supplemento* contains the following articles: S. CATANAIA, "On the solution of literal irrational equations"; G. ASCOLI, "Note on elementary geometry". There is also discussion of a number of problems. The 134th prize problem proposed by G. Ascoli is as follows:

Determine upon the side BC of a triangle ABC a point U which is such that given U' and U'' the projections of U upon AC and AB respectively, the right lines AU, BU', CU'', shall concur in a point. Demonstrate that if V and W are points on CA and AB, defined in an analogous manner, the right lines AU, BV, CW concur in a point.

The initial number of the *Proceedings of the National Academy of Sciences* contains three mathematical articles, as follows: "Recent progress in the theories of modular and formal invariants and in modular geometry," by L. E. DICKSON; "The synthesis of triad systems," by H. S. WHITE; "The ϕ-subgroup of a group of finite order," by G. A. MILLER. Professor E. H. MOORE, University of Chicago, is the mathematical editor of this journal, which is expected to appear monthly, the first issue being that for January, 1915.

It is announced that a joint session of the American Mathematical Society, the American Astronomical Society and Section A (Astronomy and Mathematics) of the American Association for the Advancement of Science will be held at the University of California on Tuesday, August 3. Addresses will be made by Professor C. J. KEYSER, of Columbia University, on "The human significance of mathematics," and by Dr. G. E. HALE, of Mount Wilson Solar Observatory, on "The work of a modern observatory."

A monument to Professor ENRICO BETTI has been erected in the suburban cemetery at Florence. The monument was designed by Professor Ristori of the Accademia di Belle Arti of Florence, and the bronze work was made under the direction of the sculptor and architect, Professor Arcangeli of Florence.

The address by Dr. FRANK SCHLESINGER, of the University of Pittsburgh, as vice-president and chairman of Section A of the American Association for the Advancement of Science at Philadelphia, appears in *Science* for January 22, the subject being "The object of astronomical and mathematical research."

School Science and Mathematics for February prints an address on "Mathematics and life—the vitalizing of secondary mathematics," delivered by Professor R. D. CARMICHAEL before the Kansas Association of Mathematics Teachers in November, together with the attendant discussion by W. T. STRATTON, of Kansas State Agricultural College.

The *Mid-West Quarterly*, a journal owned and controlled by the University of Nebraska, contains an article in the October, 1914, issue, by Professor E. W. DAVIS, on "Charles Peirce at Johns Hopkins." He was the son of the Harvard mathematician, Benjamin Peirce, and was said by Professor Sylvester to have been "a far greater mathematician than his father," though his activities in other fields were many and varied, especially in the domain of logic.

A lecture by Professor E. W. HOBSON on "John Napier and the Invention of Logarithms" has been published by the Cambridge University Press.

An article by L. C. KARPINSKI reprinted from the *Journal of Education*, Boston, on the tests by Cliff W. Stone and S. A. Courtis for determining standards of efficiency in the fundamental operations of arithmetic, is of interest in the line of recent critical examinations of the schools with a view to some definite and precise knowledge of their achievements and shortcomings.

In the September–October number of *Rendiconti del Circolo Matematico di Palermo* appears a paper on "Some properties of closed continuous curves," by Professor ARNOLD EMCH. The November–December number contains a paper on "Birational transformations of the cubic variety in four-dimensional space," by Professor VIRGIL SNYDER, and one on "Algebraic and transcendental numbers," by Professor G. N. BAUER and Dr. H. L. SLOBIN.

School and Society is the name of a new weekly journal edited by Professor J. MCKEEN CATTELL and printed by the Science Press at Lancaster, Pa. The scope of this journal is broad. To quote from the prospectus: "The journal will follow the general lines that have made *Science* of service in the sciences, coöperating with publications in special fields, aiming to become the professional journal for those engaged in the work of our lower and higher schools, and to be of interest to the wider public for whom education is of vital concern. It will emphasize the relations of education to the social order, scientific research in education and its applications, freedom of discussion, and reports and news of events of educational interest."

The committee of the New England Association of Mathematics Teachers appointed to consider the status of secondary mathematics made a preliminary report at the meeting of December 5, 1914, at which time addresses were made by Superintendent MORRISON, of New Hampshire, and Commissioner SNEDDEN, of Massachusetts. The committee has had extended correspondence with Commissioner Snedden which is likely to be published in the near future.

The committee has recently organized subcommittees on: The status of algebra; The practicability of an introductory composite course; The training of teachers of mathematics; School programs; Psychological aspects of the subject; and Mathematics for girls. A syllabus for each topic is in preparation, and correspondence with persons interested would be welcome. See Question 24 under QUESTIONS and DISCUSSIONS in this issue.

The prospectus of *The University of Chicago Science Series* states that: "The volumes of the series will differ from the discussions generally appearing in technical journals in that they will present the complete results of an experiment or series of investigations which have previously appeared only in scattered articles, if published at all. On the other hand, they will differ from detailed treatises by confining themselves to specific problems of current interest and in presenting the subject in as summary a manner and with as little technical detail as is consistent with sound method. They will be written not only for the specialist but also for the educated layman. The size of the volumes will range from fifty to one hundred and fifty pages."

The mathematical publications thus far planned or in preparation are: "Finite collineation groups," by Professor HANS BLICHFELDT, of Stanford University; and "Linear integral equations in general analysis," by Professor E. H. MOORE, of the University of Chicago. The editors of this series are Professors E. H. MOORE, J. M. COULTER, and R. A. MILLIKAN.

The meeting of the Michigan Schoolmasters' Club will be held at Ann Arbor, April 1, 2, and 3, 1915. The program of the mathematics section of the Club will consist entirely of short discussions on practical phases of the teaching of high school mathematics. On Thursday, April first, the teachers will meet at a luncheon at Newberry Hall, Ann Arbor, and the papers will be presented at the same place. The discussion on Thursday will center about the two topics: Practical applications of high school mathematics, and Correlation between mathematics and other branches and correlation of the various mathematical disciplines. Papers will be presented on the correlation between arithmetic and algebra, between algebra and geometry, and between mathematics and physics. On Friday the discussion will center about the two topics: The teaching of algebra, and The teaching of geometry.

A letter from Professor Bôcher calls attention to the fact that the holders of the Benjamin Peirce instructorships at Harvard University may not "pursue courses for higher degrees" as stated in the February number of the MONTHLY, but that as a rule the incumbents will hold the degree of doctor of philosophy. He states that the appointees for 1915–16 are Dr. EDWARD KIRCHER and Dr. G. A. PFEIFFER, both of whom received the degree of Ph.D. in 1914, the former at the University of Illinois, and the latter at Columbia University.

Mr. Bradbury, whose death was announced in the February issue, was head master of the Cambridge Latin School and not of the Boston Latin School, as stated.

4

THE

AMERICAN

MATHEMATICAL MONTHLY

VOLUME XXII APRIL, 1915 NUMBER 4

THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

V.

By FLORIAN CAJORI, Colorado College.

5. BAYLE, LEIBNIZ AND OTHER CONTINENTAL WRITERS.

Speculations by Pierre Bayle. An unusually elaborate, detailed, and critical discussion of Zeno's arguments and Aristotle's refutations is given by Pierre Bayle in the article "Zenon d'Elée," printed in his *Dictionnaire historique et critique*, 1696. Our quotations are from the English translation of the dictionary, brought out in 1710 in London. Bayle was a noted French skeptical philosopher; his article on Zeno has been widely quoted. He begins with observations on the "Arrow." Every one admits, he says, that two bodies cannot be in the same place at the same time; that "two parts of Time cannot exist together" is a theorem which "requires a little more reflection in order to apprehend it." Bayle continues:

"I will render it more obvious by an Instance. I say then that what suits Monday and Tuesday with respect to succession, suits every portion of Time whatsoever. Since then it is impossible for Monday and Tuesday to exist together, and that of necessity Monday must cease to be before Tuesday begins to be, there is no part of time whatsoever, which can co-exist with another; each must exist alone; each must begin to be, when the precedent ceaseth to be; and each must cease to be before the following can begin to exist. From whence it follows, that Time is not divisible *in infinitum*, and that the successive duration of things is composed of Moments, properly so called, each of which is simple and indivisible, perfectly distinct from time past and future, and contains no more than the present time. Those who deny this Consequence, must be given up to their Stupidity, or their want of Sincerity, or the insurmountable power of their prejudices. But if you once grant that the present time is indivisible, you will be unavoidably obliged to admit Zeno's Objection. You cannot find an instant when the Arrow leaves its place; for if you find one, it will be at the same time in that place, and yet not there. Aristotle contents himself with answering, that Zeno very falsely supposes the indivisibility of Moments."

This is Bayle's singular argument by which he tries to show that time is composed of a finite number of indivisible parts, and that, in consequence of this property of time, the difficulties of Zeno's argument that the arrow does not